

Radiation Reaction Force on a Particle.

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Abstract: The Abrahamn Lorentz radiation reaction force term, with \dot{a} , derived in text books is shown to be incomplete. We show that, with the addition of a term, the classical radiation reaction force can be generalized to the relativistic force expression. This addition is the Poynting Robertson term, seen mostly in astrophysics and usually missing from texts in electromagnetism. With this term added, it takes into account the rate of change of mass dm/dt of the particle and makes the generalization to the relativistic formula for force very straight forward.

Introduction

This paper brings together several works describing the force acting on an atom, dust particle or charge due to the radiation emitted by it, or surrounding it. It is shown that you can easily generalize the power radiated from the classical expression by Larmor to the general relativistic expression by Lienard. With our addition to the classical radiation force, we can also generalize the new force expression to the relativistic form very easily. First we introduce the Poynting Robertson force term. Then in the following section we show how this term is naturally added to the usual \dot{a} force expression. Then we give several examples of where the Poynting Robertson term naturally arises but the usual \dot{a} term is absent.

Poynting Robertson Force Term:

The radiation (pressure) force considered by Poynting and Robertson acts on a

“... spherical particle with proper mass m and world-velocity $[v^\mu]$ which scatters (whether by reflection or by absorption and re-radiation) electromagnetic radiation isotropically in all directions relative to its proper system.”

(quote from, Robertson & Noonan [1]). This definition clearly allows for a dust particle [2], a dipole oscillator [3], a 2-level atom, [4] or an electron [5-8].

All of the drag forces are of the form

$$F = -\frac{1}{c^2} Rv = -v \frac{d}{dt} \left(\frac{E}{c^2} \right) = -v \frac{dm}{dt} \quad (1)$$

where c is the velocity of light, R is the power radiated by the particle E is the energy, m is the mass and v is the velocity of the particle. This form of radiation reaction was first developed by Abraham 1903 [5] for an electron. Similar force terms arise in the LAD equation (by Lorentz, Abraham and Dirac) see Dirac [8], von Laue (1909)[6] and Pauli's book on relativity [9]. Poynting

1903 [2] derived a third of this force for a dust particle circling the sun, which was corrected by Larmor [10] in 1912 to the present force without the 1/3 factor.

There is some confusion in the literature, as to who did what and when, and which term is most important. For example, Lorentz [11], quotes Abrahams 1903 result on p31 of his Dover book, but later uses the regular Abraham–Lorentz \dot{a} force result for an electron on page 49 without the Poynting Robertson term. Most electrodynamics text books miss out the Poynting Robertson term entirely and refer to the \dot{a} force term (below) as the radiation reaction term. In the case of the point charge, the power radiated R takes the usual Larmor form

$$R = \int S_{rad} \cdot d\sigma = \frac{2}{3} \frac{e^2}{c^3} a^2 \quad (2)$$

where σ is an enclosed spherical area, a is the acceleration of the particle and S_{rad} is the time average Poynting vector of the radiation emitted from the particle.

It should be noted that the average Poynting vector is related to the energy density u via $S_{rad} = cu$, with a possible multiplicative 1/4 factor. Also note that the energy density $u = \epsilon_0 E^2$ where E is the electric field.

Typically, the radiation reaction force is accredited to Abraham and Lorentz and given by

$$F_{rad} = \frac{2}{3} \frac{e^2}{c^3} \dot{a} \quad (3)$$

where a is the acceleration of the particle (usually electron or point charge). See Jackson [12]. It should be noted that this force varies as $1/c^3$ whereas the Poynting Robertson force varies as $1/c^5$. The Poynting Robertson force might be considered a small correction to the usual radiative reaction, but without it, the generalization to relativistic expression is not obvious.

There have been mistakes in past text books, regarding who discovered what term, for example Synge's book [13] on relativity gives Larmor [14] credit for this \dot{a} force equation. However, the paper [14] only contains the power a^2 term and does not contain the \dot{a} force term.

Generalization of the Radiation Force

In this section we would like to show how one can easily generalize power radiated by a point charge to the fully relativistic form. We then generalize the classical formula for radiation reaction force with the addition of the Poynting Robertson term. Then we show how easily this force expression can be generalized to the fully relativistic form, in the same manner as the power expression was.

In teaching electromagnetism for many years you see tricks in generalizing the non-relativistic power formula to the fully covariant relativistic formula via changing $a^2 \rightarrow a^\mu a_\mu$ where \vec{a} is the classical acceleration 3–vector and $a^\mu = dv^\mu/d\tau$ is the 4–acceleration. It is quite easy to show using $x^\mu = (ct, \vec{x})$, metric $-+++$, the time dilation equation $dt = \gamma d\tau$ and $v^\mu = (\gamma c, \gamma \vec{v})$ that

$$a^\mu = \left(\gamma^4 \frac{\vec{v} \cdot \vec{a}}{c}, \gamma^4 \frac{\vec{v} \cdot \vec{a}}{c^2} \vec{v} + \gamma^2 \vec{a} \right) \quad (4)$$

Thus,

$$\begin{aligned} a^\mu a_\mu &= \gamma^6 \left[a^2 - \left(\frac{v a}{c} \right)^2 \sin^2 \theta \right] \\ &= \gamma^6 \left[a^2 - \left| \frac{\vec{v} \times \vec{a}}{c} \right|^2 \right]. \end{aligned} \quad (5)$$

This will change the non-relativistic Larmor power formula given by Eq.(2), into the relativistic power, Lienard formula which is,

$$P_{\text{REL}} = \frac{2e^2}{3c^3} \gamma^6 [a^2 - |\frac{\vec{v} \times \vec{a}}{c}|^2] . \quad (6)$$

(We have used that $dP^\mu/d\tau = a^\mu/m$, see Jackson [12].)

Now, we would like to use the same trick for the force term. We think it should generalize in the same way. First we show how to derive the full force term with the Poynting Robertson (dm/dt) addition. Take U as energy then we will equate energy loss dU/dt with force times velocity in the usual way. Starting from,

$$\begin{aligned} \vec{F} &= \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m \frac{d\vec{v}}{dt} + \frac{dm}{dt} \vec{v} \\ dU &= \vec{F} \cdot d\vec{x} + \frac{dm}{dt} \vec{v} \cdot d\vec{x} \\ \frac{dU}{dt} &= \vec{F} \cdot \vec{v} + \frac{dm}{dt} \vec{v} \cdot \vec{v} . \end{aligned} \quad (7)$$

We then time average our results assuming a cyclic pattern to the motion. We assume that the velocity and acceleration will be the same at the start and finish of the time integration, $v(t_1) = v(t_2)$ and $a(t_1) = a(t_2)$, then we equate the energy loss with the power radiated given by the Larmor formula above.

$$\int_{t_1}^{t_2} \left(\vec{F} + \frac{dm}{dt} \vec{v} \right) \cdot \vec{v} dt + \frac{2e^2}{3c^3} \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} dt = 0 \quad (8)$$

Integrate the power term by parts to find,

$$\int_{t_1}^{t_2} \left[\vec{F} - R \frac{\vec{v}}{c^2} - \frac{2e^2}{3c^3} \vec{a} \right] \cdot \vec{v} dt = 0 \quad (9)$$

Now set the square bracket equal to zero, use R as the Larmor power term, and the new field (radiation) reaction force becomes,

$$\vec{F}_{\text{RAD}} = \frac{2e^2}{3c^3} \left[\dot{\vec{a}} + \frac{\vec{v}}{c^2} a^2 \right] \quad (10)$$

This can now easily be generalized to the fully relativistic form by again setting $a^2 \rightarrow a^\mu a_\mu$, also $d\vec{a}/dt \rightarrow da_\mu/d\tau$ and $d\vec{v}/dt \rightarrow dv_\mu/d\tau$ we find,

$$F_\mu = \frac{2e^2}{3c^3} \left[\frac{da_\mu}{d\tau} + \frac{V_\mu}{c^2} a^\nu a_\nu \right] \quad (11)$$

(We have used that $dP^\mu/d\tau = a^\mu/m$ and $d^2P^\mu/d\tau^2 = \frac{1}{m} da^\mu/d\tau$, see problems section in Jackson [12].) We have used the Gaussian form here, since most of the older papers we are citing are in those units. In Gaussian $1/(4\pi\epsilon_0) = 1$ if you like to convert to S.I. units the constant term needs to be multiplied by the factor $1/(4\pi\epsilon_0)$.

To emphasize the importance and history of the Poynting Robertson force term we summarize several particle systems below, where this force term naturally arises. Note the absence of the old \dot{a} force term.

Robertson–Poynting force on a dust particle

Poynting originally describes the difference between what happens when the particle is at rest and when it is moving, so he would have derived only the second and third terms on the right of the equation below. He would have split them into a force parallel to v and one perpendicular to v (radial). The drag force is usually described as in the direction of motion, along v . Poynting originally derived only 1/3 of the force below, he neglected to use aberration of light (the change in solid angle due to motion) which would multiply a term $(1 - 2\vec{v} \cdot \hat{s}/c)$ to his considerations.

$$F = -\frac{S_{rad}A}{c} \left[\left(1 - \frac{\vec{v} \cdot \hat{s}}{c}\right) \hat{s} - \frac{\vec{v}}{c} \right] \quad (12)$$

$$= -\frac{R}{c^2} [(c - \vec{v} \cdot \hat{s})\hat{s} - \vec{v}] \quad (13)$$

where \hat{s} is a unit vector in the direction of the light beam, (Poynting vector), and \vec{v} is the velocity of the atom. The first term here is due to a stationary particle, which can be neglected. The remaining two terms should be split into components parallel to v and perpendicular to v . The drag component parallel to v is

$$\vec{F} = \frac{R}{c^2} \vec{v} . \quad (14)$$

Note that the dimensions of $R = S_{rad}A$ where A is the hemispherical area of the atom (ie. X-sectional area of atom). This is of the same form as the force above.

Page [8] also points out that a charged particle in space (no external radiation) will NOT slow down due to its own radiation alone, there must be an external source of radiation pressure to act back on the particle. It is interesting to note that Page uses electromagnetic arguments for an electron, he uses the Lienard Wiechert potentials and was the first to connect the em theory with the Poynting Robertson drag force.

Page's results were later confirmed by Larmor in a postscript to Poyntings "Collected Scientific Papers", (1920) page 757, here Larmor states,

"...the remarkable result seems to be established that an isolated body cooling in the depths of space would not change its velocity through the aether, the retardation due to the back thrust of radiation issuing from it being just compensated by increase of velocity due to momentum conserved with diminishing mass".

Larmor continues

"But for Poynting's particle describing a planetary orbit the radiation from the Sun comes in, which restores the energy lost by radiation from the particle, and so establishes again the retarding force $[-Rv/c^2]$ ".

Robertson [15] gives a nice historical review to this date and gives an equation of motion for a particle in a radiation field. A review by Burns [16] simplifies Robertson's result.

The electromagnetic Kepler problem

It is interesting to note that a dust particle [2,15] will spiral into to sun due to the drag force acting upon it by the re-radiation of the suns emissions. The exact same effect would be experienced by an electron in a Keplerian orbit about a charge Ze . [13, 17, 1]. This is described by Synge [13] as

the “electromagnetic Kepler problem”. Following Synge, he starts out with the Lorentz equation of motion for an electron as

$$m \frac{dv_\mu}{d\tau} = \frac{e}{c^2} F_{\mu\nu} v_\nu \quad (15)$$

where $v_\mu = dx_\mu/d\tau$ the proper velocity. Synge uses a 4-vector v_μ with the first three components $\gamma \vec{v}/c$ and the fourth component $i\gamma$ using $d/dt = (\gamma/c)d/d\tau$ and $x_4 = ict$. We expect the electromagnetic field tensor to have complex i 's in it multiplying the E-fields. The spatial components lead to a force equation while the time component leads to a energy loss (power) equation. The force equation leads to an equation of motion similar to an ellipse, possibly with precession of the apsides. A special case of the ellipse is the circle. The Lorentz equation of motion does not show any sign of spiraling into the center as the charged particle radiates. We must instead start with the LAD equation (Lorentz, Abraham and Dirac). M. von Laue and M. Abraham (see [9,18]) modified Lorentz's equation of motion for an electron. This equation was obtained later by Dirac [8] via a fully covariant approach. The modified equation given by Synge is

$$m \dot{v}_\mu = \frac{e}{c^2} F_{\mu\nu} v_\nu + \frac{2}{3} \frac{e^2}{c^2} (\ddot{v}_\mu - v_\mu \dot{v}_\lambda \dot{v}^\lambda) \quad (16)$$

and dot denotes differentiation with respect to (wrt) τ proper time. The last term here can be recognized as the Poynting–Robertson drag force. The temporal component of this equation gives, for the last term, the Larmor power loss due to radiation by the electron.

Rohrlich tried to use the Poynting Robertson term to allow a charge to move in a circular orbit without radiating. He took the term to be a positive force. He says in effect that there is a very small (small compared with the central force) tangential force which exactly balances the drag force and therefore allows for circular motion, [17]. Without the tangential force, the electron would also spiral into the positive charge [17,13,1]. Following the electromagnetic Kepler problem discussion by Rohrlich [17], the power of radiation emitted due to the circular motion is

$$R = \frac{2c}{3} \frac{e^2}{r^2} (\gamma^2 - 1)^2 \quad (17)$$

This is basically from the Larmor power since $(\gamma^2 - 1) = \gamma^2 v^2/c^2$ and for circular motion $a = v^2/r$ and we take $\gamma = 1$ in the final result for small velocities. The energy radiated is exactly accounted for by the positive tangential electromagnetic force.

$$\vec{F} \cdot \vec{v} = \frac{2}{3} \frac{e^2}{r^2} \gamma^2 \frac{v^2}{c} (\gamma^2 - 1) \quad (18)$$

It is clear that if we ignore the tangential force, this would cause a small change (perturbation) in the circular motion and this leads to spiral motion. The radiation rate will in zeroth order be unaffected by the change. But it can now, no longer be accounted for by the tangential force. The particle is expected to lose potential energy via radiation and fall towards the central positive charge. The kinetic energy is unchanged to first order v/c . The force term is accurate but I believe Rohrlich has taken it in the wrong direction, the tangential component is responsible for the drag and the particle will radiate and spiral into the center. The tangential force he is talking about is the Poynting Robertson drag force which causes the radiation in the first place.

Plass [19] gives a review of solutions to the Dirac equation. At the end of his lengthy article he talks about the attractive (and repulsive) coulomb force. His equation of motion for the attractive case, his eqn (162). He takes the orbit to lie in the xy-plane and writes equations for x and y time dependent motion, his eqns (163). It is not possible to obtain exact solutions to these equations although he numerically calculates curves and plots them in his Fig 10. He references Clavier [20], who finds solutions in one dimension and also finds that in 3-D a logarithmic spiral satisfies the equations at small distances from the origin.

Einstein (and Hopf): force on a 2-level atom

Einstein's 1917 work, [4] has been reproduced by Milonni [21] and a similar calculation was also performed by Boyer [22].

Einstein and Hopf [3], in (1910), derived the force acting on a dipole oscillator when it is moving through an isotropic thermal field at velocity v . Later, more clearly, Einstein 1917 derived a similar force acting on a 2-level atom when it is moving with velocity v in an isotropic thermal radiation field. Einstein calculated the power radiated by a 2-level atom by thermodynamic considerations and then using his force equation solved for the equivalent power term and set the two terms equal. Thus doing he solved the resulting first order differential eqn in energy density $\rho(\omega)$ and solved for the Planck distribution law. According to Einstein [4], in the last statement of that paper,

“ a theory (of thermal radiation) can only be regarded as justified when it is able to show that the impulses transmitted by the radiation field to matter lead to motions that are in accordance with the theory of heat.”

Einstein showed that the momentum transfer accompanying emission and absorption of radiation are consistent with statistical mechanics if the thermal radiation follows the Planck distribution. In so doing he derived a force experienced by a particle moving through a thermal field,

$$F = -\frac{\hbar\omega'}{c^2} B_{12} (N_1 - N_2) \left(\rho(\omega') - \frac{\omega'}{3} \frac{\partial \rho}{\partial \omega'} \right) v \quad (19)$$

$$= -\frac{1}{c^2} R v \quad (20)$$

$$(21)$$

where ω' is the Doppler shifted freq of the thermal radiation seen by the moving atom, B_{12} is the Einstein B-coefficient, N_1 and N_2 are the populations of levels 1 and 2 inside the 2-level atom, v is the velocity of the atom and the power radiated R becomes

$$R = \hbar\omega' B_{12} (N_1 - N_2) \left(\rho(\omega') - \frac{\omega'}{3} \frac{\partial \rho}{\partial \omega'} \right) \quad (22)$$

We note that when we take the vacuum energy density $\rho(\omega) = \hbar\omega^3/(2\pi^2c^3)$ the term

$$\left(\rho(\omega') - \frac{\omega'}{3} \frac{\partial \rho}{\partial \omega'} \right) = 0 \quad (23)$$

so there is no drag force on an atom in a vacuum. This appears to agree with Page 1918 [7].

Dirac equation of motion of an electron

The main result of Dirac's [8] paper is the equation of motion of an electron, his Eq. (24). (You can find this referenced in the problems section of Jackson [12].) The force equation reads

$$m\dot{v}_\mu - \frac{2}{3} \frac{e^2}{c^3} \ddot{v}_\mu - \frac{2}{3} \frac{e^2}{c^5} \dot{v}^2 v_\mu = ev_\nu F_{\mu\nu} \quad (24)$$

The factors of c have been written in, Dirac sets $c = 1$ throughout his paper. The 4-vector v^μ is $(c\dot{t}, \dot{x}, \dot{y}, \dot{z})$, and \dot{v}^2 term is equivalent to $\dot{v}_\mu \cdot \dot{v}^\mu \equiv (c^2 \dot{t}^2 - \dot{x}^2)$ for motion along the x-direction only. (Time derivatives are wrt. the proper time τ or s in Dirac's paper.) The first term clearly comes from the kinetic energy of the electron. The second term on the left looks like the Abraham Lorentz

radiation reaction term $F = 2e^2\dot{a}/3c^3$. The third term on the left is of the form $F = -Rv/c^2$ where R is the regular Larmor formula for power radiated by an electron of acceleration a . This clearly corresponds to the Robertson-Poynting force result. The right hand side of the Eqn. (25), involves the electromagnetic field tensor in a mixed state. $F^\nu_{\mu in} = (F_{\mu\alpha})_{in} g^{\alpha\nu}$ where for a flat space-time $g_{\alpha\beta} = g^{\alpha\beta}$ and the metric takes the form $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, and $g_{33} = -1$.

A history of the Dirac (or LAD) equation of motion is included in Rohrlich [18,23]. It can be found in Milonni's book [21] and the book by Grandy [24]. The remarkable thing about Grandy's book is that on page 204, he actually writes the drag term in the LAD equation in the Poynting-Robertson format $-Rv/c^2$ however no connection between this radiation term and the Poynting Robertson drag force is mentioned.

On the Unruh-Davies effect

Boyer[25], in (1980) noted that an electric dipole accelerated through the vacuum would see a surrounding field not quite equal to the usual Planck distribution. A correction term was needed, which turned out to be exactly the radiation reaction term given by Poynting and Robertson, see [26]. This showed that a

“ classical electric dipole oscillator accelerating though classical electromagnetic zero-point radiation responds just as would a dipole oscillator in an inertial frame in a classical thermal radiation with Planck's spectrum at temperature $T = \hbar a/2\pi ck$ ”

where T is the Unruh-Davies temperature.

Boyer [27], did a similar calculation for the spinning magnetic dipole and found a mismatch with the Planck distribution again. He later corrected the magnetic dipole work with a similar drag force to regain the Planck distribution, [28]. This latter work refers to the classical theory of spinning particles by Bhabha [29]. Boyer [28] was able to show that,

“ the departure from Planckian form is cancelled by additional terms arising in the relativistic radiative damping for the accelerating dipole. Thus the accelerating dipole behaves at equilibrium as though in an inertial frame bathed by exactly Planck's spectrum including zero-point radiation.”

Compton scattering force on an electron

Blumenthal [30], calculated the mean force due to Compton scattering on electrons with arbitrary velocity. The electron is moving along z-axis the photons are in the xy-plane making angle θ with z-axis. The force he works out for light energy density u , is eq(19c) in his paper, dP_3/dt which can be written

$$F_{\text{drag}} = -\gamma^2 \sigma_T \beta \int d\Omega u F(\eta') (1 + \beta \cos \theta)^2 \frac{\sin \theta}{1 + \eta'} \quad (25)$$

here $\eta' = \gamma u (1 + \beta \cos \theta) / mc^2$.

For small η' the function $F(\eta') = 1 - 16\eta'/5 + \dots$.

For large η' the function $F(\eta') = (3/8\eta')(\ln 2\eta' - 5/6 + \dots)$.

We shall take $\eta' \ll 1$, then $F(\eta') \approx 1$ and the equation above can be greatly simplified. Setting $\eta' \approx 0$ then

$$\begin{aligned} F_{\text{drag}} &= -\gamma^2 \sigma_T \beta \int d\Omega u (1 + \beta \cos \theta)^2 \sin \theta \\ &= -Rv/c^2 \end{aligned} \quad (26)$$

where R is the power radiated by the electron, $\beta = v/c$ and σ_T is the classical Thompson X-section (see Jackson [12])

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 \quad (27)$$

We note that $e^2/mc^2 = r_0$ is the classical electron radius. Integrating over the front half plane $0 < \theta < \pi/2$ we get

$$\begin{aligned} R &= \frac{4\pi}{3} (4\pi r_0^2) S (\pi/4 + 2\beta/3) \\ &= \pi \frac{dU}{dt} (\pi/3 + 8\beta/9) \\ &\approx \pi \frac{dU}{dt} (1 + \beta) \end{aligned} \quad (28)$$

Here we have used $S = cu$ as the Poynting vector and dU/dt as the energy loss. The resulting equation for R clearly has the correct dimensions of power and the electron radius has been used to define the spherical shell $4\pi r_0^2$ over which the Poynting vector is averaged.

If instead you choose to integrate over $0 < \theta < \pi$ then the result is

$$R = \frac{2\pi^2}{3} 4\pi r_0^2 S = \frac{2\pi^2}{3} \frac{dU}{dt} \quad (29)$$

Since we are evaluating the average Poynting vector S at a small radius r_0 from the electron, then not only do we pick up the Compton scattered light intensity but also the Bremsstrahlung of the electron no matter how small it is. So the rate of energy loss dU/dt above takes into account both the scattered radiation and any electromagnetic radiation from the electron because it is accelerating.

Conclusions

We have shown that the addition of the Poynting Robertson, mass loss term, into the standard Abraham–Lorentz radiation reaction force term (in \dot{a}) seems appropriate. Our main result of this paper is Eqn.(10) and the generalization to Eqn. (11) which agrees with the expression found in Jackson [12], written in terms of the proper momentum P^μ . The Poynting Robertson term has been missing far too long and has a long history. We have shown several instances where the term arises and is needed to explain a physical effect. We believe by inclusion of the Poynting Robertson term the non-relativistic theory can be much more easily generalized to the relativistic form and takes on the natural expected value for slower motion. Furthermore, one should note that the fully relativistic force expression agreed upon by all recent text books was originally derived by Dirac (1938) [8] using both retarded and advanced waves. Hardly any text books mention the advanced waves any longer. You need to look in an older electromagnetism text book by Panofsky and Phillips [31] to find any mention at all of advanced waves. Also Wheeler and Feynman elaborated on the advanced waves in 1945 [32] by introducing a physical mechanism, the absorber, to account for the advanced waves.

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